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# Black hole entropy from classical Liouville theory

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Abstract: In this article we compute the black hole entropy by finding a classical central charge of the Virasoro algebra of a Liouville theory using the Cardy formula. This is done by performing a dimensional reduction of the Einstein Hilbert action with the ansatz of spherical symmetry and writing the metric in conformally flat form. We obtain two coupled field equations. Using the near-horizon approximation the field equation for the conformal factor decouples. The one concerning the conformal factor is a Liouville equation, it posses the symmetry induced by a Virasoro algebra. We argue that it describes the microstates of the black hole, namely the generators of this symmetry do not change the thermodynamical properties of the black hole.

Keywords: Conformal and W Symmetry, 2D Gravity, Black Holes.

JHEP02(2003)014

## Contents



JHEP02(2003)014

## 1. Introduction

One of the most important results in black hole physics is that black holes have an entropy given by the famous Bekenstein-Hawking [1, 2, 3] formula

$$S = \frac{A_h}{4G}\,, \tag{1.1}$$

that relates the entropy of a black hole with the spatial area of its horizon $A_h$. As it is a thermodynamics relation, it would be interesting to find a microscopic interpretation of it. The Bekenstein-Hawking formula is obtained by using a semiclassical approach only, no details of a possible quantum gravity were considered. So it would be interesting to count the degrees of freedom without using details of any quantum theory of gravity. A very promising approach consists finding a central extension of an algebra of diffeomorphisms, like in the work of Strominger [4] used for the computation of the BTZ [5] black hole entropy. This approach uses the result of Brown and Henneaux [6] on the central extension of the diffeomorphism algebra that preserves the AdS structure at infinity. That is a particular case of the correspondence between $AdS_{d+1}$ spaces and conformal field theories living on the boundary [7]. But the use of local symmetries at infinity does not permit to distinguish a black hole from a star, apart from topology. Since the central role is played by the horizon of the black hole, it seems more intuitive to use near-horizon symmetries. This has been done by Carlip et al. [8, 9, 11, 12], and in other papers as well [13]. This second approach seems to involve some technical difficulties especially for the Schwarzschild black hole [14, 15, 16]. On the other hand, this way leads to difficulties in understanding the physical interpretation of the counted degrees of freedom. We shall analyze this aspect of the theory performing a dimensional reduction, as in the work of Solodukhin [9] or in [10], but with a little bit different approach. In two dimensions[1] the metric tensor is conformally flat, therefore we gain a conformally flat metric by the dimensional reduction of the four-dimensional action with spherical symmetry and by a conformal transformation. The curvature is embodied in the field that generates that transformation. At the end we obtain a theory of two scalar fields [20, 21, 22, 23]: the dilaton (i.e. the radius of the event horizon) and the Liouville field (the conformal factor of the metric) which propagate in flat space. Notice that for the thermodynamical properties of black holes the relevant aspect is the geometry of the $(r, t)$-plane, e.g. the temperature of the black hole is proportional to the surface gravity computed in the $(r, t)$-plane. This can also be seen in the euclidean approach to black hole thermodynamics [24]. Now in our model the geometry of the $(r, t)$-plane is encoded in the Liouville field. Our proposal is that this field, and its fluctuations, are responsible for the black hole entropy, and the dilaton has to be thought as fixed. On the other hand it is the geometry of the $(r, t)$-plane that is modified by the presence of a black hole, namely the singularity leads to a divergent conformal factor. Like in the work of Jackiw [25], we put our attention to the equations of motion of the conformal factor, and not on the action and the dilaton. At this stage we may consider a fixed dilaton, and consequently we may expand the dilaton potential near the horizon. We notice that the equations of motion that survive are the equation for the Liouville field, therefore we consider the action of a classical Liouville theory as the action of the black hole.

The Liouville theory has a computable classical central charge. In our dimensional-reduced model the central charge is proportional to the area of the horizon. Then the generator $L_0$ can be computed using a near-horizon approximation. Using then the Cardy [26] formula we obtain the usual Bekenstein-Hawking entropy. We use a more dynamical approach, in the sense that we consider a solution of the equation of motion, and some fluctuations around it. The reason that makes the relevant field the Liouville field has to be sought for in a possible quantum theory of gravity, but further investigations on the subject are needed. The Virasoro algebra we shall find, generating conformal diffeomorphisms, preserves the horizon and the surface gravity of the black hole with a bifurcate Killing horizon. If the black hole is extremal our construction does not hold.

The work presented here is organized in the following way: in the next section we consider the dimensionally reduced theory, and we show that this leads to a dilatonic two-dimensional theory. In section 3 we present the near-horizon approximation and its effect on the classical equations of motion. We consider only fluctuations of the conformal factor of the metric to describe the microstates of the black hole. The form of the approximated equation of motion is found to be Liouville-like. In section 4 for completeness we present the Virasoro algebra of the charges of the theory and then, in the fifth one, we compute explicitly the central charge and the generators in the case of a black hole with a bifurcate Killing horizon fixing the energy of the model equal to the energy of the black hole (namely the ADM mass). Through this way we recover the correct entropy.

[1]See also the work [17] for some aspect of 2D black hole and [18, 19] for asymptotic 2D symmetries.



JHEP02(2003)014

## 2. Dimensional reduction

In this section we want to study gravity near the horizon in the presence of spherical symmetry. This is widely known in the literature: We recall here the work concerning the black holes [20, 27, 28]. The standard procedure consists of using symmetries to reduce the dimensions of the space. Then we study the gravitational action, from a lagrangian point of view, in the presence of spherical symmetry. In particular we consider a black hole solution with a bifurcate Killing horizon. In this framework we consider only the radial degrees of freedom of the metric. As is well known, dimensional reduction leads to Liouville-like theories, or in general to dilatonic two-dimensional theories. We shall show that the equations of motion are given in terms of two fields, the dilaton $\eta$ and $\rho$ called Liouville field. Let us start with the ansatz of spherical symmetry [22]

$$ds^2 = g^{(2)}_{ab} dx^a dx^b + \Phi^2 d\Omega^2 \,, \tag{2.1}$$

where $g^{(2)}_{ab}$ is the metric of a two-dimensional spacetime, $d\Omega^2$ is the metric of a two-dimensional sphere with radius equal to one. The field $\Phi$, that depends on $x_a$, represents the radius of this sphere. We consider now the action of the four dimensional metric $g^{(4)}$,

$$I = \frac{1}{16\pi} \int \sqrt{-\gamma^{(4)}} \mathcal{R}[g^{(4)}] \tag{2.2}$$

with $G = 1$. If we integrate out the angular degrees of freedom, we obtain

$$I = \frac{1}{4} \int \sqrt{-g^{(2)}} \left( 2(\nabla\Phi)^2 + \Phi^2 \mathcal{R}[g^{(2)}] + 2 \right) . \tag{2.3}$$

This is the action of an effective two-dimensional dilatonic theory of four dimensional gravity. We notice, at this level, that the form of the two-dimensional metric is arbitrary because we have used the spherical symmetry only. This action was already studied in the literature, for example [20, 23, 29, 9, 30]. In order to put the action in a more useful form, we redefine

$$\Phi^2 = \eta \,, \qquad g^{(2)}_{ab} = \frac{1}{\sqrt{\eta}} \tilde{g}_{ab} \,, \tag{2.4}$$

obtaining the action of a dilatonic two-dimensional theory in the usual form

$$I = \frac{1}{2} \int \sqrt{-\tilde{g}} \left[ \frac{\eta}{2} \mathcal{R}[\tilde{g}] + V(\eta) \right] , \tag{2.5}$$

with $V(\eta) = 1/\sqrt{\eta}$. This equation (with general potential) describes dimensional reduced four-dimensional space, but also spaces with general dimension.[2] Before going on with the main subject of this work we shall consider briefly the classical theory. The derivation of the equations of motion from the action (2.5) is a standard procedure, they read:

$$\mathcal{R}[\tilde{g}] + 2\partial_\eta V(\eta) = 0 \,, \qquad \nabla_a \partial_b \eta - \tilde{g}_{ab} \Box_{\tilde{g}} \eta + \tilde{g}_{ab} V(\eta) = 0 \,, \tag{2.6}$$

[2] We recall that the Jackiw Teitelboim model [25, 31] (a three dimensional space) has a linear dilaton potential $V(\eta) = \eta$, the CGHS model [20] has a constant potential $V(\eta) = \lambda$.



JHEP02(2003)014

where the former equation comes from the variation of the action with respect to the field $\eta$ and the latter from the variation with respect to the two-dimensional metric $\tilde{g}$. These equations are given in terms of the dilatonic field $\eta$ and the two-dimensional metric $\tilde{g}$. The dilaton $\eta$ is related to the radius of the two-dimensional sphere by means of the former equation in (2.4) while $\tilde{g}$ encodes the geometry of the $(r,t)$-plane. The presence of a black hole modifies the geometry of the $(r,t)$-plane in a strong way. We argue that the relevant degrees of freedom for the thermodynamics behavior of the black hole are related with the geometry of the $(r,t)$-plane.

It is well known that in two dimensions the metric can always be written in the conformally flat form

$$\tilde{g}_{ab} = e^{-2\rho}\gamma_{ab} \tag{2.7}$$

where $\gamma = (-1, 1)$ is the minkowskian two-dimensional metric. Therefore the field $\rho$ describes the geometry of the $(r,t)$-plane completely. The action (2.5) becomes

$$I = \frac{1}{2}\int d^2x \left(-\partial_a\eta\partial^a\rho + V(\eta)e^{-2\rho}\right). \tag{2.8}$$

As usual, the action written above requires the constraints $\delta I/\delta g^{ab} = \alpha T_{ab} = 0$ to describe the same dynamics as the action in (2.5), indeed we have fixed the "gauge" by fixing the background metric (2.7). Now that action, with the two constraints, is an effective theory of two fields propagating in a two-dimensional, flat spacetime. After imposing (2.7), the stress tensor of (2.5) reads

$$2\,T_{ab} = -\partial_a\eta\partial_b\rho + \frac{1}{2}\partial_c\eta\partial^c\rho\gamma_{ab} - \frac{V(\eta)}{2}e^{-2\rho}\gamma_{ab} - \frac{\partial_a\partial_b\eta}{2} + \frac{\gamma_{ab}\Box_\gamma\eta}{2}. \tag{2.9}$$

Notice that it is traceless on shell. That suggests that, discarding the dilaton, and considering only the conformal factor of the metric as an effective dynamical field, one gets a conformal invariant action. For our purposes it is convenient to rewrite everything in lightcone coordinates $x^\pm = x^1 \pm x^2$. The equations of motion take the following form:

$$\partial_+\partial_-\rho - \frac{\partial_\eta V(\eta)}{4}e^{-2\rho} = 0\,, \qquad \partial_+\partial_-\eta + \frac{V(\eta)}{2}e^{-2\rho} = 0 \tag{2.10}$$

with the two constraints $T_{\pm\pm} = T_{11} + T_{22} \pm 2T_{12} = 0$:

$$\partial_+\partial_+\eta + 2\partial_+\rho\partial_+\eta = 0 \qquad \partial_-\partial_-\eta + 2\partial_-\rho\partial_-\eta = 0\,. \tag{2.11}$$

We have a system of two coupled field equations and all the geometric data of the $(r,t)$-plane are encoded in the Liouville field $\rho$, ($\gamma = (-1,1)$ for the coordinate $x^{1,2}$). Notice that, the latter equation of motion in (2.10) plus the constraints (2.11) implies the former in (2.10). It is interesting to notice that the constraints do not depend on the dilatonic potential, and so they are the same as for the theory in every dimension.

## 3. Near-horizon approximation and black holes

Up to now we have made the ansatz of spherical symmetry only. In this section we consider a spacetime with a black hole, in particular we shall analyze the consequences of its presence on the equations of motion. As pointed out above, imposing the constraints (2.11), the equations of motion (2.10) are not independent. That fact suggests that the solutions of the equations of motion which involve two fields, actually may be written in terms of a single field. Notice that even in the case of CGHS model [20] there is only one free field responsible for the whole dynamics of the black hole [28]. The standard procedure requires to fix the constraints and afterwards to integrate the equations of motion, but that is not straightforward in the case of a general potential $V(\eta)$. On the other hand, it seems more natural to study a solution of the equation of motion near the horizon of a black hole, then to consider fluctuations of one of the two fields of the effective action (2.8) around it.[3] Moreover those fluctuations shall not change the thermodynamical behavior of the black hole in question.

Therefore we discard for the moment the action, and we study the solutions of equations of motion of a two-dimensional black hole. In particular we consider some fluctuations of the field $\rho$ around the black hole solution.[4] Notice that, the dilaton takes the same value on the horizon of every non-rotating black hole with a particular temperature and entropy.[5] Whereas there are variations of the conformal factor which preserve thermodynamics describing the black hole. (For instance by multiplying the metric by a constant, the temperature and the entropy of the black hole do not change). Below we shall show those facts in details performing the near-horizon limit.

Consider a bifurcate horizon of a black hole solution of the equation of motion (2.10). As usual there is a frame of lightcone coordinates $x^{\pm}$ which covers the part of the spacetime outer of the horizon. In this coordinate patch the future/past-horizon is located at $x^{-} = +\infty/x^{+} = -\infty$, and the conformal factor tends to zero on the horizon. Let us perform the near-horizon limit of that solution. It follows, in a straightforward way, that the near-horizon limit ($x^{\pm} \to \mp\infty$) we are considering is equivalent to take the limit $\rho \to \infty$ (See section 5 for further details in the case of a near-horizon approximation of a Schwarzschild black hole by means of Rindler one.).

If we integrate the constraints (2.11) we gain the following expression:

$$\partial_{\pm}\eta = \exp\left(-2\rho + C_{\mp}(x^{\mp})\right). \tag{3.1}$$

Above $C_{\pm}$ are some finite function of $x^{\pm}$ respectively. Performing the near-horizon limit of the constraints ($x^{\pm} \to \mp\infty$ and $\rho \to \infty$), we find that the dilaton is almost constant near the horizon. Therefore, under that limit, we may consider the dilaton $\eta$ fixed at its value on the horizon $\eta_0$. In that case, one can check by inspection that, constraints (2.11) and

[3] For instance Solodukhin [9] considers only fluctuation of the dilaton $\eta$.

[4] Some other authors consider a scalar field propagating in a black hole metric finding a near-horizon CFT [32].

[5] See for instance [20].

the latter equation of motion in (2.10) are satisfied. Only the following equation for the Liouville field $\rho$ survives.

$$4\partial_+\partial_-\rho - \partial_\eta V(\eta_0)e^{-2\rho} = 0\,. \tag{3.2}$$

Hence, at least in a region near the horizon of a black hole described by the fields $\rho_B$ and $\eta_B$, there is a solution $\rho_L$ of the equation (3.2) that behaves as $\rho_B$ near the horizon. In other words, the dynamics of the system is completely described by the conformal factor $\rho$ near the horizon.

In the particular case of a four-dimensional Schwarzschild-like spacetime ($V(\eta) = 1/\sqrt{\eta}$), the "near-horizon" equation of motion (3.2) becomes:

$$\partial_+\partial_-\rho + \frac{1}{8\eta_0^{3/2}}e^{-2\rho} = 0\,. \tag{3.3}$$

This is the equation of motion of a classical Liouville theory. As is well known, the Liouville theory has a computable classical central charge. In this approximation the equation of motion (3.3) descends from an action of the form

$$I = C\int_\Sigma dx\sqrt{\hat{g}}\left(\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{\mu}{\beta^2}e^{-\beta\rho} - \frac{2}{\beta}\rho\mathcal{R}[\hat{g}]\right), \tag{3.4}$$

with $\beta = 2$ and $\mu = -2V'(\eta_0)$ which, in the case of $V(\eta) = 1/\sqrt{\eta}$, is $\mu = 1/\eta_0^{3/2}$. We stress that we have not derived this action (3.4) from the gravitational one (2.8), but we have written an action that leads to the expected equation of motion (3.3). Jackiw in his well known paper [25] follows the same procedure. Moreover our theory has to be thought as an effective theory of the black hole. Perhaps, a more satisfactory description of that assumption has to be sought in a possible quantum theory.

The factor $C$ of that effective theory differs from that presented above. In the next we have to fix it in order to have the energy equal to the mass of the black hole.[6] To check this effective model of the bifurcate black hole, we shall compute the entropy and compare it with the thermodynamical one, finding a total agreement. Notice that we are studying a theory on a fixed background metric $\hat{g}$. Since we have extracted the conformal factor $\rho$, we choose a flat background metric $\hat{g} = \gamma = (-1, 1)$. Therefore the term containing $\mathcal{R}[\hat{g}]$ vanishes. It is well known that such a theory possesses a classical central charge, and to find it we have to know the stress-energy tensor $T$. In lightcone coordinates it has the following non vanishing components

$$T_{\pm\pm} = C\left(\partial_\pm\rho\partial_\pm\rho + \frac{2}{\beta}\partial_\pm\partial_\pm\rho\right), \tag{3.5}$$

notice that the last term arises from the variation of the action with respect to the metric before fixing the gauge, it is called improved stress-energy tensor. As is well known, the total classical central charge [25] of this theory is

$$c = 12C\frac{4}{\beta^2}\,. \tag{3.6}$$

[6] Another procedure, concerning micro-canonical action of general relativity, can be found in [33].

Notice that the value of the central charge computed above depends on the normalization constant $C$ of the stress tensor. Moreover, both the central charge and the stress tensor do not depend on either $\mu$ and the particular value of $\eta_0$. In this section we have derived the central charge of the gravitational action performing a certain near-horizon approximation and considering the conformal factor as the only dynamical field. Notice that this central charge is peculiar to the black hole. We are considering all the fluctuations of the conformal factor as degrees of freedom of the classical theory. At this level some physical question arises, for example it would be interesting to understand how to interpret the fluctuations from a geometrical point of view.

## 4. The Virasoro algebra

In this section we try to construct the full Virasoro algebra, and then, using the Cardy formula [26], we compute the entropy of the effective theory describing the black hole. We have already noticed that variations of the conformal factor correspond to conformal transformations of the action of the black hole. As usual, the generators $L_n$ of the conformal transformations arise from the stress tensor $T_{\pm\pm}$, and are computed smearing the stress tensor by some function.

$$L_n^{\pm}(x^{\pm}) = \int_S dx^{\pm}\xi^{\pm}T_{\pm\pm}\,, \tag{4.1}$$

where $S$ is the set that contains the $x^{\pm}$. The $L_n$ are called the charges (Noether current) of the symmetry in question. A general charge is composed of two parts corresponding to a right and a left moving plane waves $\xi^{+}(x^{+}) + \xi^{-}(x^{-})$ on the fixed space with metric $\gamma$. The solution of the Liouville equation (3.4) is $\rho = \rho^{+}(x^{+}) + \rho^{-}(x^{-})$, the $T_{++}$ and $T_{--}$ depend respectively only on $x^{+}$ and $x^{-}$. The usual Poisson bracket of the charges (4.1) satisfy the following relations[7]

$$\{L_n^{\pm}, L_m^{\pm}\}_{\mathcal{PB}} = L^{\pm}[\xi_n, \xi_m] + \frac{c^{\pm}}{12}\Delta(\xi_n, \xi_m)\,, \qquad \Delta = \int dx^{\pm}\left[\xi_n^{\pm}\partial_{\pm}^3\xi_m^{\pm} - \xi_m^{\pm}\partial_{\pm}^3\xi_n^{\pm}\right], \tag{4.2}$$

where the bracket $[,]$ is the usual Lie bracket of fields $\xi_n = \xi_n^{+} + \xi_n^{-}$. Notice that we have two classes of Virasoro generators $L_n^{\pm}$ they form two independent algebras. The total central charge is given by the sum of the central charges of the two algebras

$$c = c^{+} + c^{-}\,. \tag{4.3}$$

In order to find a Virasoro algebra we must specify the form of $\xi^{\pm}$. We set

$$\xi_n^{\pm} = \frac{\ell}{2\pi}\exp\left(-\frac{i\,n\,2\pi x^{\pm}}{\ell}\right) \tag{4.4}$$

where the factor $\ell/2\pi$ is necessary in order that the vector fields $\xi^{\pm}$ closes with respect to the Lie brackets and $\ell$ is a cutoff parameter. With that prescription the generators become

$$L_n^{\pm} = -\frac{\ell}{2\pi}\int_{-\ell/2}^{\ell/2} dx^{\pm}\,\exp\left(\frac{-i\,n\,2\pi x^{\pm}}{\ell}\right)T_{\pm\pm}\,. \tag{4.5}$$

[7] For further details see [25] for instance.

The $L_n^\pm$ of that form satisfy (4.2). By shifting $L_0$, it is possible to put it in the well known Virasoro form:

$$\{L_n^\pm, L_m^\pm\}_{\mathcal{PB}} = i(n-m)L_{n+m}^\pm + i\frac{c^\pm}{12}(n^3-n)\delta_{n+m}\,. \tag{4.6}$$

This is true if the constant $C$ in $T_{\pm\pm}$ equals one. In our case $C$ is not one therefore we shall rescale the Poisson brackets, as explained later, in order to close the algebra. Before ending this section we want to spend some words on the meaning of the cutoff parameter $\ell$. The reason for its presence is that we want to find a countable set of $L_n^\pm$ and this can only be done by smearing the stress energy tensor with the Fourier modes ($\xi_n$) on a compact space, and then considering the limit as the cutoff goes to infinity without changing the algebra.

## 5. Counting the microstates

Let us consider an explicit solution of the equations of motion, and compute the associated $T_{\pm\pm}$. We shall give a possible description of the microstates responsible for the thermodynamics behavior of the black hole. Therefore we consider a spacetime with a bifurcate Killing horizon. In general, the metric of a bifurcate Killing horizon generated by a black hole has the form

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega^2\,, \tag{5.1}$$

where $d\Omega^2$ is the metric of the unit sphere. If we consider the near-horizon approximation $r \sim r_0$, the function $A(r) = A'(r_0)(r-r_0)$ and $A'(r_0) = 2\kappa$, the metric of the $(r,t)$-plane becomes that of a Rindler space time with $r = r_0 + \kappa y^2/2$:

$$ds^2 = -\kappa^2 y^2 dt^2 + dy^2\,. \tag{5.2}$$

It can be brought in conformal form

$$ds^2 = -dx^+ dx^- \exp(x^+ - x^-) \tag{5.3}$$

where the new coordinates are $x^- = \kappa t - \log y$ and $x^+ = \kappa t + \log y$. This metric, as expected, is conformally flat and the conformal factor is

$$-2\rho = x^+ - x^- = 2\log y\,. \tag{5.4}$$

The past horizon is located at $x^+ = -\infty$ whereas the future horizon is located at $x^- = +\infty$. In these coordinates, the fields $\xi_n$ become

$$\xi_n^\pm \partial_\pm = \xi_n^\pm\left(\kappa\partial_t \pm y\partial_y\right). \tag{5.5}$$

It would by nice if these fields, thought as generators of symmetry transformations, preserve the temperature and the entropy of the black hole in question. To check this, we recall the expression of the temperature $T_h$ of the black hole,

$$T_h = \frac{1}{2\pi}\frac{\partial_y N}{\sigma}(r_0) \tag{5.6}$$

where $N$ is called the lapse function of the metric and $\sigma$ is the factor of the spatial part $ds^2 = -N^2 dt + \sigma^2 dy^2$, for the metric (5.2) $N = \kappa y$ and $\sigma = 1$.



JHEP02(2003)014

$\partial_y N/\sigma = \kappa$ is the surface gravity of the black hole which is proportional to the temperature. We want to check if the variation of the temperature, namely the surface gravity is zero under a transformation generated by $\xi_n^{\pm}$. Moreover, we want to check if the horizon of the black hole is preserved, we recall that the horizon is located where the lapse function vanishes. If its variation vanishes, $\delta N = 0$, the variation of the spatial area of the horizon, which is proportional to the entropy of the black hole, vanishes too.

We have the following Lie derivatives

$$\mathcal{L}_{\xi_n^{\pm}} N(r_0) = 0\,, \qquad \mathcal{L}_{\xi_n^{\pm}} \frac{\partial_y N}{\sigma}(r_0) = \kappa \xi_n^{\pm} \left[ \left( in\frac{2\pi}{\ell} + 1 \right)^2 - \left( in\frac{2\pi}{\ell} + 1 \right) \right]. \tag{5.7}$$

The variation of the lapse is always zero, and the variation of the surface gravity is zero in the limit of large $\ell$. Our fluctuations of the conformal factor of the metric, described by the field $\rho$, in the limit of large $\ell$ preserve the temperature and the entropy of the black hole, this means that we are computing a class of metrics compatible with the same black hole (fully described by its temperature and entropy). However the conformal transformations in question is physically meaningful only when the cutoff $\ell$ tends to infinity. (The walls of the box tend to infinity.)

Notice that we have a classical Liouville field theory on a fixed flat background. We shall fix the energy of this effective model equal to the energy of the black hole, namely its ADM mass. In that way it is possible to fix the normalization of the action. The ADM mass counts the energy of a spacetime endowed with a black hole as seen by an observer located at infinity. It sounds natural that an effective theory describing that black hole should have the same energy. We stress that the stress tensor $T_{\pm\pm}$ we are considering is also an effective stress tensor and not the gravitational one. On the other hand the Rindler metric is a near-horizon approximation of any black hole with a bifurcate Killing horizon. As just pointed out, to fix the energy of the theory, we fix the constant $C$ of the stress tensor (3.5) as follows

$$C = \frac{\kappa A_h}{2\pi\ell}\,, \tag{5.8}$$

where $A_h$ is the spatial area of the horizon and $\kappa$ is the surface gravity. The energy of the system, computed by that factor $C$, equals the mass of the black hole $M_B$, namely since $T_{11} = T_{22}$

$$\int_{-\ell/\sqrt{2}}^{\ell/\sqrt{2}} T_{11} dx_2 = \frac{1}{4}\int_{-\ell/2}^{\ell/2} T_{++} + T_{--} dx^+ + \frac{1}{4}\int_{-\ell/2}^{\ell/2} T_{++} + T_{--} dx^- = \frac{\kappa A_h}{8\pi} = M_B\,. \tag{5.9}$$

As the result does not depend on the value of $\ell$, it holds when the cutoff parameter tends to infinity. Moreover if we rescale the coordinates by a factor $1/a$, $\tilde{x} = x/a$, the energy of the theory does not change. In particular, $\tilde{\partial}_{\pm}\tilde{\rho}(\tilde{x}) = \partial_{\pm}\rho(x)$ and $\tilde{\partial}_{\pm}\tilde{\partial}_{\pm}\tilde{\rho}(\tilde{x}) = a\partial_{\pm}\partial_{\pm}\rho(x)$. Moreover the vectors $\tilde{\xi}^{\pm}(\tilde{x})/a\tilde{\partial}_{\pm} = \xi^{\pm}(x)\partial_{\pm}$. Discarding the "tildes", the new charges $L_n^{\pm}$ become

$$L_n^{\pm} = \frac{A_h \kappa}{2\pi\ell}\int_{-\ell/2a}^{\ell/2a} dx^{\pm}\, a\left(\frac{\ell}{2a\pi}\right) \exp\left(-i\frac{2a\pi}{\ell}nx^{\pm}\right)\left(\partial_{\pm}\rho\partial_{\pm}\rho + \frac{1}{a}\partial_{\pm}\partial_{\pm}\rho\right). \tag{5.10}$$

Since $C$ is not equal to one, we have to rescale the Poisson brackets in order to have a Virasoro algebra. Notice that the charges suited above satisfy the following Virasoro algebra

$$\{L_n^{\pm}, L_m^{\pm}\} = i(n-m)L_{n+m}^{\pm} + i\frac{c^{\pm}}{12}n^3\delta_{n+m}\,, \tag{5.11}$$

where the brackets computed above are the usual Poisson bracket $\{\cdot,\cdot\}_{\mathcal{PB}}$ of (4.6) rescaled by a factor $-1/C$, namely $\{\cdot,\cdot\} = -(2\pi\ell)/(A_h\kappa)\{\cdot,\cdot\}_{\mathcal{PB}}$. In order to explicitly write the influence of $\kappa$ of the (5.2) in the charges we choose $a = \kappa$, getting the background metric $ds^2 = -\kappa^2 dx^+ dx^-$. Notice that the time $(x^+ + x^-)/2$ is equal to the Rindler time $t$ of (5.2) and the euclidean Rindler time $\tau = -i(x^+ + x^-)/2$ is periodic with period $2\pi/\kappa$. In the new coordinates the generator $L_0^{\pm}$ becomes

$$L_0^+ = \frac{A_h \ell}{16\pi^2} \tag{5.12}$$

and the corresponding central charge reads

$$c^+ = \frac{3A_h}{2\ell}\,. \tag{5.13}$$

We are eventually able to compute the entropy of this theory using the logarithm of the Cardy formula

$$S = 2\pi\sqrt{\frac{c^+ L_0^+}{6}} = \frac{A_h}{4} \tag{5.14}$$

That is nothing but the Bekenstein-Hawking entropy associated with the bifurcate Killing horizon (5.1) for every value of the cutoff parameter $\ell$.

The charges $L_n^{\pm}$ diverge as $\ell$, and the central charge tends to zero as $1/\ell$ for large value of $\ell$, therefore their product do not depend on $\ell$. So we are able to compute the black hole entropy by using the Cardy formula. We recall that a zero central charge was also found in [34, 35]. We have used only one copy of the Virasoro algebra, that corresponding to the future horizon, because the event horizon of the physical black hole is given by the future horizon.

## 6. Discussion and open problems

We have studied the conformal factor of a dimensional reduced theory in the presence of a black hole with a bifurcate Killing horizon. In this case we have shown that near the horizon, it is described by a Liouville theory, which does not modify the thermodynamical properties of the black hole. This has a classical central charge and, in fact, we have shown that it can be used to compute the entropy of the black hole in question. If the black hole is extremal it does not posses a bifurcate Killing horizon and in this case the geometry of the $(r,t)$-plane is not that of a Rindler space but it is an $AdS_2$ space. In this case our approximation does not hold because the exponential term in the equation (3.3) is positive, and so it is not a Liouville equation. On the other hand, the temperature of an extremal black hole is zero.



JHEP02(2003)014

We stress the fact that we have made no assumptions on a specific quantum gravity model but used only the classical near-horizon structure of the black hole. That approach differs from the already studied models because the central role is played by the conformal field $\rho$ and not by the dilaton $\eta$. The effective conformal theory found above describes the micro-canonical theory responsible for the entropy of the black hole, indeed we have fixed the value of $\int T_{tt}dr$ as its mass $M_B$. We have found the correct thermodynamical entropy even if the central charge is zero if computed on the global space, and even if the fundamental mode $L_0^+$ diverges as the cutoff parameter $\ell$ tends to infinity. The charges, if computed on the whole space time, correspond to conformal transformations that do not change the thermodynamical properties of the black hole. On the other hand we have made no assumptions on the boundary condition we have to impose to the conformal factor. Perhaps some other correction should be searched in that direction.


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JHEP02(2003)014